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Localization and Computation in an Approximation of Eigenvalues

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Abstract. In this note we consider the problem of localization and approximation of eigenvalues of operators on infinite dimensional Banach and Hilbert spaces. This problem has been studied for operators of finite rank but it is seldom investigated in the infinite dimensional case. The eigenvalues of an operator (between infinite dimensional vector spaces) can be positioned in different parts of the spectrum of the operator, even it is not necessary to be isolated points in the spectrum. Also, an isolated point in the spectrum is not necessary an eigenvalue. One method that we can apply is using Weyl's theorem for an operator, which asserts that every point outside the Weyl spectrum is an isolated eigenvalue.

1. Introduction

Many problems in applied linear algebra can be reduced to eigenvalue problems, that is, for a given linear transformation T, we need to know its eigenvalues. Let X be a Banach space over the complex field $\mathbb{C}$, we say $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if there exists a non-zero vector $x \in X$ such that $Tx = \lambda x$ (or equivalently $(T - \lambda I)x = 0$). The corresponding non-zero vector x is called an *eigenvector* associated with the eigenvalue λ. The problem of determining the eigenvalues has been widely studied when X is finite-dimensional and, provided the dimension is reasonable for computations, exact eigenvalues can be found. On the other hand, when X is infinite-dimensional the exact eigenvalues can rarely be found even when they are isolated points of the spectrum. Thus, we have to use some kind of approximation for such point in the spectrum, and generalize this process for the case of spectral sets.

Let $B(X)$ denote the algebra of bounded linear transformations (equivalently, operators) of X into itself. $T \in B(X)$ is said to be *invertible* if there exists an operator $S \in B(X)$ such that $TS = ST = I$, where I denotes the identity in $B(X)$; in this case we write $T^{-1} := S$. Let $\sigma(T)$ denote the (usual) *spectrum* of T, i.e.

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

and $\rho(T) := \mathbb{C} \setminus \sigma(T)$ denote the resolvent set of T. The set of all eigenvalues of T will be denoted by $\sigma_p(T)$. The spectrum is a nonempty compact set of $\mathbb{C}$, but it may happen that $\sigma_p(T)$ is empty (see Example 3.1 below).

Recall $T \in B(X)$ is a *Fredholm operator* if $\dim N(T) < \infty$ and $\operatorname{codim} R(T)(= \dim X/R(T)) < \infty$, where $N(T)$ denotes the kernel of T and $R(T)$ the range of T. If $\dim N(T) = \operatorname{codim} R(T) < \infty$ then we say T is a *Weyl operator*.

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For $T \in B(X)$,

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}$$

is the *Fredholm* (*essential*) *spectrum* of T, and

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$$

is the *Weyl spectrum* of T.

Let $\Lambda \subset \sigma(T)$ be such that Λ is open and closed in the relative topology of $\sigma(T)$, then we say that Λ is a *spectral set* for T.

An *elementary Cauchy domain* is a bounded open connected subset of $\mathbb{C}$ whose boundary is the union of a finite number of nonintersecting Jordan curves. A finite union of elementary Cauchy domains with disjoint closures is called a *Cauchy domain*.

Let D be a Cauchy domain. If each involved Jordan curve is oriented in such a way that the points in D lie to the left side when the curve is traced out, then the oriented boundary is called a *Cauchy contour*.

Let C be a Cauchy contour, then we define $\text{int}(C) = D$ for the interior of C and $\text{ext}(C) = \mathbb{C} \setminus (D \cup C)$ for the exterior. Let $E, \tilde{E} \subset \mathbb{C}$ be such that $E \subset \text{int}(C)$ and $\tilde{E} \subset \text{ext}(C)$, then we say that C separates E from $\tilde{E}$. The set of all Cauchy contours separating Λ from $\sigma(T) \setminus \Lambda$ will be denoted by $\mathcal{C}(T, \Lambda)$.

For $z \in \rho(T)$,

$$\mathcal{R}(T, z) := (T - zI)^{-1}$$

is called the *resolvent operator* of T at z.

For a spectral set Λ for $T \in B(X)$ and $C \in \mathcal{C}(T, \Lambda)$, define

$$P(T, \Lambda) = -\frac{1}{2\pi i} \int_C \mathcal{R}(T, z) dz.$$

It is well known that $P(T, \Lambda)$ is a bounded projection of T and Λ, and it does not depend on the particular choosing of $C \in \mathcal{C}(T, \Lambda)$. Let $M = R(P)$ and $N = N(P)$, then $\sigma(T|_M) = \Lambda$ and $\sigma(T|_N) = \sigma(T) \setminus \Lambda$ (for more details see [14, pg. 178]).

If $\lambda \in \sigma(T)$ is an isolated point, it is clear that $\{\lambda\}$ is a spectral set for T; let $P(T, \lambda)$ denote the projection related to T and $\{\lambda\}$. For $\lambda \in \sigma(T)$ we say that it is a *Riesz point* (or finite rank pole) of T if λ is an isolated eigenvalue of T of finite algebraic multiplicity, i.e. $\dim P(T, \lambda)(X) < \infty$. For an isolated eigenvalue λ of T, we say that it has *finite geometric multiplicity* if $\dim N(T - \lambda I) < \infty$. Throughout this note $\pi_0(T)$ will stand for the set of Riesz points of T and $\pi_{00}(T)$ for the set of isolated eigenvalues of T of finite geometric multiplicity. It is known that $\pi_0(T) = \text{iso}\sigma(T) \setminus \sigma_w(T)$, where $\text{iso}\sigma(T)$ stands for the isolated points of $\sigma(T)$, and $\pi_0(T) \subset \pi_{00}(T)$ (see [12], [13]).

2. Continuity of spectrum

In this paper, for bounded linear operators T and T_n, $n \in \mathbb{N}$, we will say that T_n converges to T, in notation $T_n \to T$, if T_n converges to T in norm, i.e. $\|T_n - T\| \to 0$ as $n \to \infty$. We can observe the spectrum of an operator as a mapping from $B(X)$ to the set of all compact nonempty subsets of the complex plane. In this way, we can ask if this mapping is continuous with respect to the norm metric in $B(X)$ and the Hausdorff metric in the second space. For the sake of better understanding, we will use the following approach:

Let $B_\epsilon(\lambda)$ be the open ball centered at λ with radius ϵ. If (τ_n) is a sequence of compact subsets of $\mathbb{C}$, then the *limit inferior*, in notation $\liminf \tau_n$, is

$$\liminf \tau_n := \{\lambda \in \mathbb{C} \,:\, \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, B_\epsilon(\lambda) \cap \tau_n \neq \emptyset\},$$

and the *limit superior*, in notation $\limsup \tau_n$, is

$$\limsup \tau_n := \{\lambda \in \mathbb{C} \,:\, \forall \epsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n > n_0 \text{ such that } B_\epsilon(\lambda) \cap \tau_n \neq \emptyset\}.$$

If $\liminf \tau_n = \limsup \tau_n$, then $\lim \tau_n$ is said to exist and it is equal to this common limit.

Let p be a mapping on $B(X)$ whose values are compact subsets of $\mathbb{C}$. A mapping p is *upper* (*lower*) *semicontinuous* at T, if for every sequence $\{T_n\} \subset B(X)$ such that $T_n \to T$ then $\limsup p(T_n) \subset p(T)$ ($p(T) \subset \liminf p(T_n)$). If p is both upper and lower semicontinuous at T, then p is *continuous* at T and in this case we write $\lim p(T_n) = p(T)$.

Example 2.1. In general, the spectrum is not a continuous mapping in the previous sense.

Let H be the Hilbert space $\ell_2(\mathbb{Z})$ with the usual norm. For $x \in X$, let

$$T(x)(k) = \begin{cases} x(k+1), & k \neq -1; \\ 0, & k = -1. \end{cases} \qquad T_n(x)(k) = \begin{cases} x(k+1), & k \neq -1; \\ \frac{x(0)}{n} & k = -1. \end{cases}$$

Then $\|T_n - T\| \to 0$, $\sigma(T) = \{x \in X : |x| \leq 1\}$, $\sigma(T_n) = \{x \in X : |x| = 1\}$ and, consequently, $\sigma(T) \neq \lim \sigma(T_n)$.

The simple fact that the set of all invertible operators in the Banach algebra $B(X)$ is open implies that the spectrum is an upper semicontinuous mapping. Indeed, let $T_n \to T$ (in norm) and $\lambda \in \limsup \sigma(T_n)$. Then there exists a sequence of positive integer $\{n_k\}$ and a sequence of points $\{\lambda_{n_k}\}$, $\lambda_{n_k} \in \sigma(T_{n_k})$, such that $\lambda_{n_k} \to \lambda$. Hence, the sequence $T_{n_k} - \lambda_{n_k}$ of non-invertible elements converges in norm to $T - \lambda$, which implies $\lambda \in \sigma(T)$ and this give us the following (see [16]):

Theorem 2.2. *The spectrum is an upper semicontinuous mapping in the algebra of bounded linear operators over a Banach space X.*

By previous theorem for continuity of the spectrum at $T \in B(X)$ it is sufficient (and necessary) to prove lower semi-continuity, i.e. that $\sigma(T) \subset \liminf \sigma(T_n)$. From [18, Proposition 2.3] follows that $\sigma(T) \setminus \sigma_{ap}(T) \subset \liminf \sigma(T_n)$, where $\sigma_{ap}(T)$ denote the approximate point spectrum ($\lambda \in \sigma_{ap}(T)$ if and only if $T - \lambda$ is not bounded below). Hence, for continuity of the spectrum we must have $\sigma_{ap}(T) \subset \liminf \sigma(T_n)$. The next theorem partially gives an answer for it, but we need some more terminology.

For non-zero $T \in B(X)$, the reduced minimum modulus is defined by

$$\gamma(T) =: \inf_{x \notin N(T)} \frac{\|Tx\|}{\operatorname{dist}(x, N(T))}.$$

Theorem 2.3. *Let $\{T_n\}$ be a sequence in $B(X)$ that converges in norm to $T \in B(X)$. If γ is uniformly bounded below on $\{T_n - \lambda \,:\, \lambda \in \sigma_{ap}(T),\ n \in \mathbb{N}\}$, then*

$$\sigma(T) = \lim \sigma(T_n).$$

Proof. By Theorem 2.2 (and comment below it) it is enough to show that $\sigma_{ap}(T) \subset \liminf \sigma(T_n)$. Let $\delta > 0$ such that $\gamma(T_n - \lambda) > \delta \geq 0$, for every positive integer n and every $\lambda \in \sigma_{ap}(T)$. Suppose that there exists a point $\lambda \in \sigma_{ap}(T)$ such that $\lambda \notin \liminf \sigma(T_n)$. Since $\lambda \in \sigma_{ap}(T)$ there exists a sequence $\{x_m\}$ of norm one vectors such that $\|(T - \lambda)x_m\| \to 0$. Without loss of generality, we can suppose that $\lambda \notin \sigma(T_n)$, for every positive integer n, and consequently

$$0 \leq \delta < \gamma(T_n - \lambda) = \inf_{\|x\|=1} \|(T_n - \lambda)x\| \leq \|(T_n - \lambda)x_m\|,$$

for every positive integers n and m. But

$$\|(T_n - \lambda)x_m\| \leq \|(T_n - T)x_m\| + \|(T - \lambda)x_m\| \longrightarrow 0, \qquad n, m \to \infty,$$

giving us a contradiction. □

Remark 2.4. The condition that $\gamma(T_n - \lambda)$ is uniformly bounded below for some sequence $\{T_n\}$ that converges in norm to T, with $\lambda \in \sigma_{ap}(T)$, is not such scarce in $B(X)$. By $\mathcal{G}(X)$ we denote the class of all operators S in $B(X)$ such that $(S - \lambda)^{-1}$ is normaloid for all $\lambda \notin \sigma(S)$, i.e. $r((S - \lambda)^{-1}) = \|(S - \lambda)^{-1}\|$, where $r(S)$ is the spectral radius of S. It is known that classes of all normal and hyponormal operators belong to the class $\mathcal{G}(X)$ (for definitions of such operators see next section).

Let $\{S_n\} \subset \mathcal{G}(X)$ such that $\lambda \notin \liminf \sigma(S_n)$. Then, there exists a $\delta > 0$ such that $d(\sigma(S_n), \lambda) > \delta$ (in general it is true for some subsequence $\{S_{n_k}\}$ of $\{S_n\}$ but, without loss of generality, we can suppose that it is true for this sequence). Then, by [15, pg. 203], $\gamma(S_n - \lambda) = \|(S_n - \lambda)^{-1}\|^{-1}$ and we have

$$\begin{aligned}\gamma(S_n - \lambda) = \|(S_n - \lambda)^{-1}\|^{-1} = (r(S_n - \lambda)^{-1})^{-1} &= (\max\{|\mu| \,:\, \mu \in \sigma((S_n - \lambda)^{-1})\})^{-1}\\ &= d(\sigma(S_n), \lambda) > \delta\end{aligned}$$

(for more details see [11, proof of Lemma 2]).

Corollary 2.5. *Let the sequence $\{T_n\} \subset \mathcal{G}(X)$ converge in norm to $T \in B(X)$. Then*

$$\sigma(T) = \lim \sigma(T_n).$$

Proof. Follows from Theorem 2.3 and Remark 2.4. □

3. Localization of poles and Weyl's theorem

In the theory of spectral approximation, it is known that it is easier to approximate a (simple) pole of an operator than an eigenvalue of (one) finite geometric multiplicity (for more details see [2]). Hence we prefer that, for some operator $T \in B(X)$, $\pi_{00}(T) = \pi_0(T)$. In general, we have $\pi_0(T) \subset \pi_{00}(T)$. To see this we can observe any $\lambda \in \operatorname{iso} \sigma(T)$, and relationship between the subspace $M = R(P(T, \lambda))$ and the null space of $T - \lambda$ or to the null space of some power of $T - \lambda$. Using the fact $\cup_{n=1}^{\infty} N[(T - \lambda)^n] \subset M$ (for more details see [1], [10], [13]) we get the inclusion. The answer on the question when equality holds we can find trough Browder theorem for an operator. We say that an operator obeys *Browder's theorem* if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T).$$

The stronger version of this is called *Weyl's theorem*; for $T \in B(X)$ we say Weyl's theorem holds if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

It is known that Weyl's theorem implies Browder's theorem and Browder's theorem with condition $\pi_0(T) = \pi_{00}(T)$ implies Weyl's theorem for operator T (for more details see [8, Theorem 8.3.1]).

Both of these theorems for an operator $T \in B(X)$ guarantees us that every complex number outside the Weyl spectrum of T are isolated eigenvalues. Moreover, Weyl's theorem for T tells us that such a point is a pole of finite algebraic multiplicity. Indeed, the very first work in this way is the paper of H. Weyl (see [20]). He studied the spectra of a self adjoint operator T on a Hilbert space and discovered that $\lambda \in \sigma(T + K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$.

Suppose that λ is an isolated point of $\sigma(T)$. In the case when X is not finite dimensional, it is not necessary that such point is an eigenvalue. To see this, let $T \in B(X)$ be any injective quasinilpotent operator. Then $0 \in \operatorname{iso} \sigma(T)$ and it is not eigenvalue of T. Now, we give a example of such operator T.

Example 3.1. Let ℓ^2 be the Hilbert space of all square summable sequences, and let $T \in B(\ell^2)$ the operator defined by

$$T(x_1, x_2, x_3, \dots) := (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots).$$

It is easy to see that $(T - \lambda I)$ is injective for all $\lambda \in \mathbb{C}$ and $\sigma(T) = \{0\}$. Indeed, a straight calculation give us $\|T^n\| = \frac{1}{n!}$, hence $\|T^n\|^{1/n} \to 0$ as $n \to \infty$, which implies $\sigma(T) = \{0\}$.

On the other hand, there exist huge classes of linear operators for which the isolated points of $\sigma(T)$ are eigenvalues of T. Such operators are called *isoloid*.

Definition 3.2. Let H be a Hilbert space, then $T \in B(H)$ is said to be

- normal if $\|T^*\| = \|T\|$;
- hyponormal if $\|T^*\| \leq \|T\|$;
- p-hyponormal, $0 < p \leq 1$, if $\|T^*\|^{2p} \leq \|T\|^{2p}$;
- M-hyponormal if there exists an $M \geq 1$ such that $\|(T - \lambda I)^* x\| \leq M\|(T - \lambda I)x\|$ for all $\lambda \in \mathbb{C}$ and $x \in H$;
- totally paranormal if $\|(T - \lambda)x\|^2 \leq \|(T - \lambda)^2 x\|$ for each unit vector $x \in H$ and any complex number λ.

It is known that a normal operator is hyponormal, a 1-hyponormal operator is hyponormal and any p-hyponormal operator T is is paranormal (see [4],[6],[17]). We shall denote by $\mathcal{P}$ the union of all these classes of operators.

Theorem 3.3. *Let $T \in \mathcal{P}$. Then every isolated point of $\sigma(T)$ is an eigenvalue (indeed a simple pole) of T.*

Proof. For the case of p-hyponormal operator see [5, Lemma 3.4], for M-hyponormal consult [3, Theorem 3] and for totally paranormal [19, Proposition 2.4]. □

In almost every paper that covers topics about Weyl's type theorems or spectral continuity in some special class of operators (see [3], [4], [5], [6], [11]) it is supposed that the sequence of operators and its limit operator belong to the same class. In the next theorem we do not require that the limit of operators is in the same class.

Theorem 3.4. *Let $\{T_n\} \subset B(H)$ be a sequence of operators such that T_n are in $\mathcal{P}$ and suppose T_n converges in norm to an operator T. Then Weyl's theorem holds for T.*

Proof. The proof will be done trough several steps.

STEP I: $\lim \sigma(T_n) = \sigma(T)$.

Without loss of generality we can suppose that all sequence $\{T_n\}$ belongs to only one subclass of $\mathcal{P}$. Then $\lim \sigma(T_n) = \sigma(T)$ (in the case of p-hyponormal operators see [5], totally paranormal operators case follows by Corollary 2.5 and Remark 2.4, M-hyponormal case by [7]).

STEP II: $\pi_{00}(T) = \pi_0(T)$.

Let $\lambda \in \pi_{00}(T)$. By continuity of the spectrum at T, $\lambda \in \liminf \sigma(T_n)$, i.e. there exists a (sub)sequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \in \sigma(T_{n_k})$ and $\lambda_{n_k} \to \lambda$. Since $\lambda \in \operatorname{iso} \sigma(T)$, then λ_{n_k} is an isolated point in $\sigma(T_{n_k})$, for n_k big enough, and, by Theorem 3.3, $\lambda_{n_k} \in \pi_0(T_{n_k})$. Let C be a Cauchy contour that separates λ from the rest of $\sigma(T)$. Then C separates λ_{n_k} from rest of $\sigma(T_{n_k})$. Let

$$P = P(T, \lambda) = -\frac{1}{2\pi i} \int_C \mathcal{R}(T, z) dz$$

and

$$P_{n_k} = P(T_{n_k}, \lambda_{n_k}) = -\frac{1}{2\pi i} \int_C \mathcal{R}(T_{n_k}, z) dz.$$

Then, $\dim R(P_{n_k}) = 1$, $P_{n_k} \to P$, which implies $\lambda \in \pi_0(T)$.

STEP III: $\lim \sigma_w(T_n) = \sigma_w(T)$.

Suppose the contrary, then there exists $\lambda \in \sigma_w(T) \setminus \liminf \sigma_w(T_n)$. By STEP I, $\lambda \in \liminf \sigma(T_n)$ and consequently, there exists a (sub)sequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \in \sigma(T_{n_k}) \setminus \sigma_w(T_{n_k})(= \pi_0(T_{n_k}))$ and $\lambda_{n_k} \to \lambda$. By previous STEP, we have that $\lambda \in \pi_0(T)$ that implies $\lambda \notin \sigma_w(T)$. A contradiction.

STEP IV: Weyl's theorem holds for T.

To complete the proof, it is enough to show that $\operatorname{acc} \sigma(T) \subseteq \sigma_w(T)$, the rest follows by [8, Pg. 115 and Thm. 8.4.5]. Suppose that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then, by Steps I and III, $\lambda \in \lim \sigma(T_n) \setminus \lim \sigma_w(T_n)$ and, consequently, there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \in \sigma(T_n) \setminus \sigma_w(T_n)(= \pi_0(T_n))$ and $\lambda_n \to \lambda$. By the proof of STEP II, we have that $\lambda \in \pi_0(T) \subset \operatorname{iso} \sigma(T)$. □

Remark 3.5. For $\lambda \in \operatorname{iso}\sigma(T)$ and any sequence $\{T_n\} \subset B(X)$ that converges in norm to T, we have that $\lambda \in \liminf \sigma(T_n)$ (see [16]). But, if $\lambda \in \pi_0(T)$ we have even more, then there exists a sequence $\lambda_n \in \pi_0(T_n)$ such that $\lambda_n \to \lambda$ and $\dim P(T,\lambda) = \dim P(T_n, \lambda_n)$ (see [12, Proposition 50.2]).

An eigenvalue $\lambda \in \pi_0(T)$ is called a *simple pole* if $\dim P(T,\lambda) = 1$, or equivalently, the algebraic multiplicity of a simple pole λ is 1. It is known that λ is a simple pole of T if and only if $X = N(T - \lambda I) \oplus R(T - \lambda I)$.

Let λ be a simple pole of T and $\{T_n\}$ any sequence of bounded linear operators that converge in norm to T. Then, from Remark 3.5, for any $\epsilon > 0$ there is a positive integer n_0 such that, for each $n \geq n_0$, we have a unique $\lambda_n \in \sigma(T_n)$ satisfying $|\lambda - \lambda_n| < \epsilon$. Also, λ_n is a simple pole of T_n and $\lambda_n \to \lambda$.

Theorem 3.6. *Let $\{T_n : n = 0, 1, 2, \ldots\} \subset \mathcal{P}$, $T_n \to T_0$(in norm) and $\lambda_0 \in \operatorname{iso}\sigma(T_0)$. Then for n large enough, there exists unique $\lambda_n \in \pi_0(T_n)$ such that*

$$|\lambda_n - \lambda_0| \leq \frac{1}{1 - \|P_n - P_0\|}\|T_n - T_0\|.$$

Proof. Let $P_n = P(T_n, \lambda_n)$ and $M_n = R(P_n)$, $n = 0, 1, 2\ldots$. By Theorem 3.3 and Remark 3.5 we have that, for n large enough, $\dim M_0 = 1 = \dim M_n$. Moreover, by [1, Theorem 3.74], $M_0 = N(T_0 - \lambda_0 I)$ and $M_n = N(T_n - \lambda_n I)$. Also, since $T_n \to T_0$, then $P_n \to P_0$. Additionally, all of P_n, $n = 0, 1, 2, \ldots$, are self-adjoint (see [9, 2.3 Applications (a)]) and $\|P_n\| = 1$.

Let φ_0 be an unit eigenvector of T_0 corresponding to λ_0 and, for n large enough (say $n > n_0$), let φ_n be an unit eigenvector of T_n corresponding to λ_n.

Let $Q_n : M_n \to M_0$ be defined by $Q_n(x) = P_0(x)$, for every $x \in M_n$. Then Q_n is bijective. We split the proof of this fact in several steps.

I. For n large enough, $I - (P_n - P_0)$ and $I - (P_0 - P_n)$ are invertible. Since $\|P_n - P_0\| \to 0$, without loss of generality, we can suppose that $\|P_0 - P_n\| \leq 1$, for every $n > n_0$, and consequently the operators $I - (P_n - P_0)$ and $I - (P_0 - P_n)$ are invertible.

II. Q_n is injective. Let $x \in M_n$ be such that $Q_n(x) = 0$. Then

$$(I - (P_n - P_0))x = x - P_n x + P_0 x = 0.$$

Since $I - (P_n - P_0)$ is invertible, it follows that $x = 0$.

III. Q_n is surjective. Let $y \in M_0$ be an arbitrary vector. Then, by invertibility of the operator $I - (P_n - P_0)$, there exists $x \in H$ such that $y = (I - (P_n - P_0))x$. Let $z = P_n x (= y - x + P_0 x)$. Then

$$P_0 z = P_0(y - x + P_0 x) = P_0 y - P_0 x + P_0 x = P_0 y = y.$$

Hence, Q_n is surjective.

Moreover, for every $x \in M_n$, we have

$$\|x\| - \|P_0 x\| \leq \|x - P_0 x\| = \|P_n x - P_0 P_n x\| \leq \|P_n - P_0\|\|x\|,$$

and consequently $\|Q_n^{-1}\| \leq \frac{1}{1 - \|P_n - P_0\|}$.

By invertibility of Q_n it is easy to see that $P_0\varphi_n (= Q_n(\varphi_n))$ is a nonzero element of $N(T_0 - \lambda_0 I)$, and consequently, $P_0\varphi_n$ is an eigenvector of T_0 corresponding to λ_0. Also, $\varphi_n = Q_n^{-1} P_0 \varphi_n$. Thus, for all large n,

$$\lambda_n \varphi_n = \lambda_n Q_n^{-1} P_0 \varphi_n = (Q_n^{-1} P_0 T_n)\varphi_n$$

and

$$\lambda_0 \varphi_n = Q_n^{-1} \lambda_0 P_0 \varphi_n = Q_n^{-1}(T_0(P_0\varphi_n)) = (Q_n^{-1} P_0 T_0)\varphi_n.$$

Since $\|\varphi_n\| = 1$, we have

$$|\lambda_n - \lambda_0| = \|\lambda_n\varphi_n - \lambda_0\varphi_n\| = \|(Q_n^{-1}P_0)(T_n - T_0)\varphi_n\| \leq \|Q_n^{-1}\|\|P_0\|\|T_n - T_0\| \leq$$

$$\leq \frac{1}{1 - \|P_n - P_0\|}\|T_n - T_0\|.$$

□

A spectral set Λ is called a *spectral set of finite type* if the corresponding spectral subspace $P(T,\Lambda)(X)$ is finite dimensional.

It is well known (see, for example, [2, Theorem 2.2]) that if Λ is a nonempty spectral set for T and P is the corresponding spectral projection, then Λ is a spectral set of finite type for T if and only if Λ consists of a finite number of spectral values of T, each of which is of finite type. Furthermore, if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and P_j denotes the spectral projection corresponding to T and λ_j, $j = 1, \ldots, r$, then

$$P = P_1 + \cdots + P_r, \qquad P_i P_j = 0,\ i \neq j,$$

and $\mathrm{rank}(P) = m_1 + \cdots m_r$, where m_j is the algebraic multiplicity of λ_j, $j = 1, \ldots, r$.

Example 3.7. Let X be an n-dimensional linear space and $T \in B(X)$ be a linear operator with matrix representation $A \in \mathbb{C}^{n\times n}$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be the spectrum of T (or A). For $P_i = P(T, \lambda_i)$, we have

$$P_1 + P_2 + \cdots + P_r = I \text{ and } \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_r = X.$$

The subspaces $\mathcal{P}_i$ are T-invariant, and the matrix representation of the operator T obtained using the basis of the subspaces $\mathcal{P}_i$ is the Jordan canonical form of A.

References

[1] P. Aiena, *Fredholm and Local Spectral Theory, with applications to Multipliers*, Kluwer Acad. Publisher, 2004.
[2] M. Ahues, A. Largillier and B.V. Limaye, *Spectral computations for bounded operators*, Chapman&Hall/CRC, 2001.
[3] S. C. Arora and R. Kumar, *M-hyponormal operators*, Yokohama Mathematical Journal **28** (1980), 41–44.
[4] S.L. Campbell and B.C. Gupta, *On k-quasihyponormal operators*, Math. Japon. **23** (1978-79), 185–189.
[5] S.V. Djordjević and B.P. Duggal, *Weyls theorems and continuity of spectra in the class of p-hyponormal operators*, Studia Math. **143 (1)** (2000), 23–32.
[6] B.P. Duggal and S.V. Djordjević, *Generalized Weyl's theorem for a class of operators satisfying a norm condition*, Math. Proc. Royal Irish Acad., **104 A** (2004), 75-81.
[7] B.P. Duggal, *Paranormal and (p,k)-quasihyponormal operators: Spectral continuity*, Rend. Circolo Mat. Di Palermo Serie II, **57** (2007), 57–64.
[8] B.P. Duggal, *SVEP, Browder and Weyl theorems*, Topicos de Teoria de la Approximacion **III**, BUAP, Dirección de Fomento Editorial, (2009), 107–146.
[9] B.P. Duggal and S.V. Djordjević, *Self-adjoint Riesz projections for Hilbert space operators*, Topicos de Teoria de la Approximacion **IV**, BUAP, Dirección de Fomento Editorial, (2012), 11–19.
[10] I. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of Linear Operators*, Vol. 1, Birkhauser Verlag, Basel, 1990.
[11] I.S. Hwang and W.Y. Lee, *The spectrums is continuous on the set of p-hyponormal operators*, Math. Z. **235** (2000), 151-157.
[12] H. G. Heuser, Functional Analysis, John Wiley and Sons, 1982.
[13] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Math. Anal. **6**(1958), 261-322.
[14] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1976.
[15] K.B. Laursen and M.M. Neumann, An Introduction to Local Spectral Theory, Oxford Science Publications, 2000.
[16] J. D. Newburgh, *The variation of spectra*, Duke Math. J. **18** (1951), 165–176.
[17] M. Radjabalipour, *On majorization and normality of operators*, Proc. Amer. Math. Soc. **62** (1977), 105–110.
[18] S. Sánchez-Perales and S.V. Djordjević, *Spectral continuity of operator matrices*, Bul. Korean Math. Soc. **48** (2011), 1261-1270.
[19] C. Schmoeger, *On totally paranormal operators*, Bull. Austral. Math. Soc. **66** (2002), 425–441.
[20] H. Weyl, *Uber beschrankte quadratiche Formen, deren Differenz vollsteig ist.* Rend. Circ. Mat. Palermo **27** (1909), 373-392.